

# An Approach to a Discrete Quantum Mechanics in Closed Riemannian Spaces

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(Z. Naturforsch. **28 a**, 1613–1619 [1973]; received 19 October 1973)

*To Professor Konrad Bleuler on the occasion of his 60th birthday*

Quantum mechanics is contemplated in closed Riemannian spaces where the momentum spectrum is discrete. The theory formulated in this case is equivalent to an exclusively discrete scheme; continuity occurs only statistically. The operators corresponding to the dynamical variables are determined and the uncertainty relation is examined. Agreement is obtained in the local limit with quantum mechanics as formulated in flat space within the experimental accuracy.

## I. Introduction

The ideas on space, time and their relation to physics were fundamentally modified in the last century by Riemann<sup>1</sup>. Riemann recognized that geometrical and physical laws are complementary in the description of nature, so that to a large family of geometries the appropriate physical laws can be formulated. Preference is in general given to the simplest possible combination of geometry and physics. Developing such general geometries Riemann recognized that even topology and physics are interrelated. He envisaged unlimited finite spaces such as for example the 3-sphere. His ideas embraced also topology in the widest sense contemplating even physical spaces of discrete points; he considered such discrete spaces superior to the continuum in the way that they inhibit a natural measure whereas that of the continuum can only be established with the help of further physical laws.

The general theory of relativity constitutes a mere triumph of the first mentioned ideas on continuum geometry. Einstein discovered the gravitational law which determines the optimum geometry – not only of space but to a large extent even that of the space time continuum. His theory admits unlimited finite spaces one example of which, the Einstein universe<sup>2</sup> is indeed a 3-sphere which is perpendicular to the time axis. Such a universe forms also the space-time background in the present work. The Einstein universe was later modified for empirical and theoretical reasons to take account of

the observed recession of distant objects and to secure stability. We consider it however adequate within the scope of the present investigation.

Riemann's ideas on discreteness experienced a triumph hitherto only in the physics of matter where it was contemplated by atomists since ancient times. What is the situation in other domains of the description of nature? Schrödinger discusses the intricacies of the continuum in his booklet *Science and Humanism*<sup>3</sup>. The discreteness which quantum physics has achieved for a part of the spectrum of physical values is considered by Schrödinger a progress over the classical continuum. He strives to extend the discreteness by contemplating solutions of the wave equations in closed spaces: "Wave mechanics imposes an a priori reason for space to be closed; for then and only then are its proper modes discontinuous and provide an adequate description of the observed atomicity of matter and light<sup>4-7</sup>". Remarkably we have only been able to find the mentioned publications on the discrete spectrum of the wave equations in closed universes by Schrödinger but no treatment of a general quantum mechanical formalism underlying it. The same is true for the numerous personal discussions on the subject to which he has given us the occasion. We do not know therefore of any way by which Schrödinger would have tried to overcome the formal difficulties associated with a quantum mechanics in closed spaces where exclusively discrete eigenvalues of the momentum operator occur. Related difficulties in the formulation of the commutation relations are for example mentioned by Weyl<sup>8</sup>.

Section II of the present work deals with the formulation of the commutation relations first in a

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one dimensional model of a closed space and then in the Einstein universe. Section III treats as an example the harmonic oscillator in the one dimensional model. The modified uncertainty relation is analyzed in section IV. The results agree numerically with those of open space quantum mechanics in best approximation as long as only phenomena within spatial domains small compared to the radius of the universe are considered\*. The structure of open and closed space quantum mechanics is however not the same even if the radius of the universe tends to infinity. The momentum variables in the closed universe are in a way preferred to the position variables. They have a discrete spectrum and they allow formulation of the theory in a discrete scheme. Uncertainty of the momentum results only from spatial limitation below the range of the whole universe. The position operator admits a continuous spectrum but it is interpreted merely as an auxiliary entity which actually belongs to the tangent space of the background space-time manifold. The discrete scheme introduced here is thus fundamentally different from that of lattice spaces (see esp.<sup>10</sup> and <sup>11</sup>) where the space continuum itself is quantized. The manifestation of the continuum in the present context is a statistical one\*\*.

## II. Presentation of the Modified Formalism

We consider a closed three dimensional Riemannian space for example that of Einsteins spherical universe which we introduced briefly in Section I. We deal here exclusively with one particle states in the space of the universe. Such a space can be embedded in a four dimensional Euclidian space and its points are determined by the equation:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2. \quad (1)$$

The radius  $R$  is very large, at least  $10^{27}$  cm. In order to be able to take advantage of the symmetry of the space we introduce global coordinates; for example in our spherical space:

$$\begin{aligned} x_1 &= R \sin \chi \sin \vartheta \cos \varphi, \\ x_2 &= R \sin \chi \sin \vartheta \sin \varphi, \end{aligned}$$

\* Dautcourt investigates the detectability of the topological structure of the universe<sup>9</sup>. He considers the wave equation in closed universes and finds by approximation of the solutions that the discrete structure of the levels is too fine for detection. The author seems not to have had access to Schrödingers work on exact solutions of wave equations in closed universes. We here assume beforehand that the discrete structure is not detectable and concentrate on the modification of the formalism resulting from spatial closedness.

$$x_3 = R \sin \chi \cos \vartheta,$$

$$x_4 = R \cos \chi$$

with the line element:

$$ds^2 = -R^2 \{ d\chi^2 + \sin^2 \chi (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \} + c^2 dt^2. \quad (1a)$$

The coordinate lines are then necessarily closed. Functions which depend only on the points of this space have to be periodic in such coordinates. This must hold also for the solutions of the wave equation generalized to such a space; the solution for a given time are periodic in the space coordinates. Such boundary condition result in a discrete eigenvalue spectrum. We try to establish a quantum mechanical formalism in our space; if we use for this purpose a space of functions in the coordinate representation the properties of our physical space leads us to the remarkable requirement that *only periodic functions be admissible*. We require furthermore that within the limited domain of the space of the universe which is accessible to us ordinary quantum mechanics should hold in a good approximation. One should thus expect that in a system of geodesic coordinates in the neighbourhood of the origin there exist position operators  $Q$  acting on state functions  $\Psi(x)$  in the  $x$ -representation:  $Q \Psi(x) = x \Psi(x)$  and momentum operators  $P$ :

$$P \Psi(x) = i \frac{\partial}{\partial x} \Psi(x). \quad (2)$$

Globally this requirement cannot be upheld for even if the coordinate line is a geodesic  $Q \Psi(x)$  as defined above does not fulfill our periodicity requirement. One may try to make it periodic by introducing a jump at some point\*\*\* but  $Q \Psi$  is then not differentiable at this point and the momentum operator in the above form is not defined for it. The admission of a function with a jump for  $Q \Psi$  causes also difficulties to the physical interpretation, especially with the uncertainties.

Related difficulties occur whenever a spatially periodic motion is treated quantum mechanically

\*\* A more detailed discussion of the present ideas will appear in a forthcoming article in the Bulletin de l'Académie Royale Belge.

\*\*\* The periodicity is restored this way in a work by Susskind and Glogower<sup>12</sup> in order to remedy the situation in case of the angle-angular momentum system. I am grateful to Prof. Whippman of the University of Helsinki for this reference.

and the angle variables are chosen as curvilinear coordinates. Classically the angle variable increases proportional to the time whereas in quantum mechanics the eigenvalues must not exceed  $2\pi$  because of the periodicity requirement chosen. The expectation values can thus formally not agree with the classical result. The conjugate action variables have only discrete eigenvalues and the commutation relations can then according to the argument of Section I not be of the required form. An argument sometimes heard in this connection is that such variables are not appropriately chosen for the quantum mechanical problem" and should be replaced by another pair of conjugate variables which fulfill the required commutation rules. Such an escape is however no more available in our present case of a closed universe where the analogous pair of variables which in flat space fulfill the commutation rules have also the properties of the angle action type.

The way we go to be able to treat the problem is to choose a *periodic* function of the position variable  $Q$  and to use the well defined commutation rules between the  $P$  conjugate to  $Q$  and this *function of  $Q$*  to solve our dynamical problem. To simplify the demonstration of this method we reduce for the moment our closed spherical space from three to one dimension — that means to a circle of radius  $R$ ; we choose units such that  $\hbar = c = 1$  and that  $R$  has the value one. The wave equation of a particle of vanishing rest mass is then:

$$\partial^2 \psi / \partial t^2 - \partial^2 \psi / \partial x^2 = 0 \quad (3)$$

and a complete set of solutions which fulfill the boundary condition are:

$$\Psi_n(x, t) = e^{in(t-x)} \quad n \text{ integer).} \quad (3a)$$

These solutions are eigenstates of the momentum operator  $P$  with eigenvalues  $n$ . We have no difficulty to determine the commutation relations of  $P$  with an operator  $\sin Q$  defined by:

$$\begin{aligned} \sin Q \Psi(x, t) &= \sin x \Psi(x, t), \\ [P, \sin Q] &= i \cos Q. \end{aligned} \quad (4)$$

A matrix representation of these relations for  $P$  diagonal is:

$$P_{mn} = n \delta_{mn} \quad (\sin Q)_{mn} = \frac{1}{2} i (\delta_{m,n+1} - \delta_{m,n-1}), \quad (4a)$$

$$([P, \sin Q])_{mn} = i (\cos Q)_{mn} = \frac{1}{2} i (\delta_{m,n+1} + \delta_{m,n-1}). \quad (4b)$$

One finds immediately the relations:

$$\sin^2 Q = \frac{1}{4} (2 \delta_{m,n} - \delta_{m,n-2} - \delta_{m,n+2}), \quad (4c)$$

$$([P, \sin^2 Q])_{mn} = \frac{1}{2} (\delta_{m,n=m+2} - \delta_{m,n=m-2}), \quad (4d)$$

$$([\sin Q, P^2])_{mn} = -\frac{1}{2} i (\delta_{m,n+1} \cdot (2m-1) + \delta_{m,n-1} \cdot (2m+1)). \quad (4e)$$

The operators  $\sin Q$  and  $\cos Q$  have thus the matrix form:

$$\sin Q = \frac{1}{2} i \begin{bmatrix} 0, & -1, & 0, & 0 \\ 1, & 0, & -1, & 0 \\ 0, & 1, & 0, & -1 \\ 0, & 0, & 1, & 0 \end{bmatrix} \quad \cos Q = \frac{1}{2} \begin{bmatrix} 0, & 1, & 0, & 0 \\ 1, & 0, & 1, & 0 \\ 0, & 1, & 0, & 1 \\ 0, & 0, & 1, & 0 \end{bmatrix}. \quad (4f)$$

Let us denote the eigenvectors of  $P$  by  $|n\rangle : P|n\rangle = n|n\rangle$  one can express the eigenvectors of  $\sin Q$  in terms of the  $|n\rangle$ :

$$\alpha_1 = \sum_{n=-\infty}^{\infty} e^{inx} |n\rangle \quad \text{and} \quad \alpha_2 = \sum_{n=-\infty}^{\infty} e^{in(x+\pi)} |n\rangle \quad (4g)$$

they correspond to the two different states both of which have the eigenvalue  $\sin x$  \*.

We have now established commutation rules by which we are able to solve dynamical problems. This is demonstrated in the next section where the analog of the harmonic oscillator is treated. We may now raise the question for the true position operator. We remember that we wanted to establish a scheme that within the extensions of our domain of experience reproduces the results of quantum mechanics in adequate approximation. The spatial extensions of the solar system in our units lies below  $10^{-10}$ . The operator  $\sin Q$  should therefore be an excellent approximation for the position operator in the neighbourhood of the origin. We write henceforth frequently:

$$\sin Q = Q(0) \quad (4h)$$

to indicate that it is considered as our position operator near the origin. Equation (4b) which gives the form of  $[P, Q(0)]$  shows us that the expectation values of this operator formed with eigen-

\* Operators  $\sin Q$  and  $\cos Q$  have been introduced by Susskind and Glogower (Ref. <sup>12</sup>) in order to obtain a phase variable conjugate to the energy of the harmonic oscillator. Their operators are different from the present because  $P$

has a spectrum different from that of the occupation number operator. The case of the rotator which formally resembles the present problem, is treated there differently (see previous footnote).

states of  $P$  have not the value  $i$  obtained in ordinary quantum mechanics but rather the value zero. Our extreme localization in the universe prevents us however from preparing anything comparable to these purest states. The states we are able to prepare and to measure are a rather broad superposition of these purest momentum states:  $\psi = \sum_{n=-\infty}^{\infty} a_n |n\rangle$  where

states with neighbouring eigenvalues cannot be distinguished and are expected to have in average very nearly the same amplitudes and phases. The expectation values formed with such states  $\Psi$  agree under this assumption as easily seen with those of ordinary quantum mechanics. The position operator  $Q(x_0)$  in the neighbourhood of a point  $x_0$  is of the form:

$$Q_{mn}(x_0) = \frac{1}{2} i (e^{ix_0} \delta_{m,n+1} - e^{-ix_0} \delta_{m,n-1}). \quad (4j)$$

$Q(0)$  and  $Q(x_0)$  commute as long as we deal with an infinite spectrum of the momentum operator. This assumption of an arbitrarily large momentum value is however not compatible with a closed space the curvature of which is caused by the gravitational field of the matter distribution in it. If one simply cuts off the momentum spectrum at values  $n = \pm N$  we obtain instead:

$$([Q(0), Q(x_0)])_{mn} = \frac{1}{2} i (\delta_{m,N} \delta_{n,N} - \delta_{m,-N} \delta_{n,-N}) \sin x_0 \quad (5)$$

that means only terms at the extremities of the matrix  $\neq 0$ . The noncommutability then shows up only at the states of highest momentum which may not be attainable. The effect of the noncommutability on the position states may however entail a modification of the topology of space which we aim to analyze in a subsequent paper. The possibility to achieve finite matrix representations and a resulting modification of the topology of space was one of the chief motives for the present work \*\*.

We close this section by formulating the position operators  $Q_\alpha(0)$  on the three dimensional surface which forms the space of Einsteins universe. This surface is invariant with respect to the group  $O_4$  of four dimensional rotations and the six generators of the group are related to the momentum and angular momentum operators  $N_\alpha, M_\alpha$  ( $\alpha = 1, 2, 3$ ).

Setting  $R=1$  again, in the four dimensional Euclidian space into which we have embedded the spherical surface these operators are:

$$M_\alpha = -i(x_\beta \partial/\partial x_\gamma - x_\gamma \partial/\partial x_\beta) \\ (\alpha, \beta, \gamma) = (1, 2, 3) \text{ or cyclic,} \quad (6a)$$

$$N_\alpha = -i(x_4 \partial/\partial x_\alpha - x_\alpha \partial/\partial x_4) \\ (\alpha = 1, 2, 3); \quad (6b)$$

we follow in this presentation Schrödinger<sup>7</sup>.

At the point  $x_1 = x_2 = x_3 = 0, x_4 = R$  the  $M_\alpha$  are the angular momentum operators and the  $N_\alpha/R$  are the momentum operators; but at the point  $x_2 = x_3 = x_4 = 0, x_1 = R$ :  $M_2/R, M_3/R, -N_1/R$  are momentum operators and  $M_1, N_2, N_3$  are angular momentum operators. A corresponding superposition of the operators has to be chosen at other points. In the neighbourhood of the point  $x_4 = R, x_1 = x_2 = x_3 = 0$  which we chose as the origin of our coordinate system the role of  $M_\alpha, N_\alpha$  is approximately the same as at the origin. The commutation rules are:

$$[N_\alpha, N_\beta] = [M_\alpha, M_\beta] = i M_\gamma,$$

$$[N_\alpha, M_\beta] = [M_\alpha, N_\beta] = i N_\gamma,$$

$$[N_\alpha, M_\alpha] = 0 \quad (\alpha, \beta, \gamma) = (1, 2, 3) \text{ or cyclic.} \quad (6c)$$

Casimir operators are:

$$\sum_{\alpha=1}^3 M_\alpha N_\alpha \quad \text{and} \quad K = \sum_{\alpha=1}^3 (M_\alpha^2 + N_\alpha^2). \quad (6d)$$

The commutation rules imply that only one pair ( $M_\alpha, N_\alpha$ ) can have sharp eigenvalues, in which case the expectation values of all the other components vanish<sup>7</sup>. The possible eigenvalues are best determined by the combinations:

$$\xi_\alpha = M_\alpha + N_\alpha \quad (\text{eigenv.} -\alpha \dots \alpha)$$

and

$$\eta_\alpha = M_\alpha - N_\alpha \quad (\text{eigenv.} -\beta \dots \beta)$$

$$\alpha, \beta \text{ integer or half integer.} \quad (6e)$$

The components of the  $\xi$  commute with those of the  $\eta$  and each of them has the same spectrum of eigenvalues as the angular momentum operators in ordinary quantum mechanics. Thus  $O_4 = O_3 \times O_3$  accordingly:

$$K = 2 \sum_{\alpha} (\xi_\alpha^2 + \eta_\alpha^2) \\ M \cdot N = \sum_{\alpha} (\xi_\alpha^2 - \eta_\alpha^2)$$

\*\* If  $N$  is very large one may work with the finite matrices just as if they were infinite and obtain results most of which do not differ measurably from those of ordinary quantum mechanics. The situation may be compared to boundary conditions of a very large volume of a solid which

do not affect the physics in the interior appreciably. I thank Prof. Højgaard Jensen for pointing this out and for supplying me with literature on the notion of phonon position in solids<sup>13</sup>.



$K$  has eigenvalues  $2[\alpha(\alpha+1) + \beta(\beta+1)]$ , (6 f)

$M \cdot N$  has eigenvalues  $(\alpha - \beta)(\alpha + \beta + 1)$ . (6 g)

$M \cdot N = 0$  for a tensor representation. For our representations we have thus with  $\alpha + \beta = n$ :

$$\left. \begin{array}{l} K \\ M^2 \\ N^2 \\ M_3 \end{array} \right\} \text{ has } \left\{ \begin{array}{l} n(n+2) \quad (n=0, 1, 2, 3 \dots), \quad (6 h) \\ L(L+1), \quad (6 j) \\ n(n+2) - L(L+1), \quad (6 k) \\ (-L, -L+1, \dots, L). \quad (6 l) \end{array} \right.$$

$(1/R^2)K$  is the Laplace operator generalized to the surface of the sphere. In polar coordinates (1 a):

$$K \Psi = \frac{1}{\sin^2 \chi} \left\{ \frac{\partial}{\partial \chi} \sin^2 \chi \frac{\partial \Psi}{\partial \chi} + \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \sin \vartheta \frac{\partial \Psi}{\partial \vartheta} + \frac{1}{\sin^2 \vartheta} \frac{\partial^2 \Psi}{\partial \varphi^2} \right\} \quad (6 m)$$

it contains  $\vartheta, \varphi$  in form of the ordinary spherical operators. Our wave equation becomes in three dimensions:

$$(1/R^2)K \Psi + E^2 \Psi = 0, \quad (6 n)$$

$E^2$  assumes the values  $n(n+2)$  in units where  $R=1$ .

Solutions of these equation which fulfill the boundary conditions and which correspond to given eigenvalues  $n, L$  have been quoted by Schrödinger<sup>7</sup>; they are of the form:

$$\Psi_{nL}^m = S_{nL}^* Y_L^m(\vartheta, \varphi). \quad (6 o)$$

The  $Y_L^m$  are spherical harmonics and

$$S_n^L = (1-x^2)^{(L/2)} \frac{d^L}{dx^L} S_n^0 \quad (6 p)$$

and  $S_n^0 = (x^2-1)^{-\frac{1}{2}} d^n (x^2-1)^{n+\frac{1}{2}} / dx^n$  are Gegenbauer polynomials in  $x = \cos \chi$ . (6 q)

One has in particular:

$$S_1^0 = 3 \cos \chi \quad \text{and} \quad S_1^1 = 3 \sin \chi. \quad (6 r)$$

We are now in the position to define the analog of our position operator  $Q(0)$  in the neighbourhood of the origin  $\chi=0$  of our coordinate system on the hypersphere:

$$\begin{aligned} Q_1(0) \Psi &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} S_1^1 (Y_1^{-1} - Y_1^1) \Psi, \\ Q_2(0) \Psi &= i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} S_1^1 (Y_1^1 + Y_1^{-1}) \Psi, \\ Q_3(0) \Psi &= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} S_1^1 Y_1^0 \Psi. \end{aligned} \quad (7)$$

This rule allows to obtain the matrix elements in the case of a representation in terms of the  $\psi_{nm}^L$  straightforwardly because of the orthogonality of these functions. The situation in the three dimensional case will be analyzed in detail in a subsequent article.

### III. Applications in the One-dimensional Case

We want to show that with the formalism developed in Section II we can indeed reproduce the results of ordinary quantum mechanics in the neighbourhood of the origin. We do this first by an explicit example, the one dimensional oscillator and try to apply then general considerations. The oscillator in the one-dimensional universe must have a potential fulfilling also the boundary condition. We choose:  $V = a^2 \sin^2 x/2$  which has a maximum at  $x=\pi$ . The non-relativistic Schroedinger equation becomes in our units:

$$y'' + 2m(E - a^2 \sin^2 x/2)y = 0. \quad (1 a)$$

This can be transformed with  $u = \frac{1}{2}(x + \pi)$  into:

$$d^2 y / du^2 + 8m[(E - a^2/2) - \frac{1}{2}a^2 \cos 2u]y = 0 \quad (1 b)$$

which is the standard form of the Mathieu equation<sup>14</sup>. We need solutions with the period  $\pi$  in  $u$  to fulfill the boundary conditions. The value of the constant  $a$  of the oscillator potential has to be assumed gigantic in order to be observable due to our tiny units. The same is true for the rest mass  $m$ . In such a case the eigenvalues of the Mathieu functions of the above periodicity are approximated by:

$$8m(E - a^2/2) = -8ma^2/2 + 2wa\sqrt{2m} - (w^2 + 1)/8 - O(a^{-2}). \quad (2)$$

Here  $w = 2n + 1$  (Ref. <sup>14</sup>) thus

$$E = (n + 1/2)a\sqrt{2m} - (w^2 + 1)/64m - O(a^{-2}) \quad (2 a)$$

which for all reasonable values of  $n$  and the mass  $m$  is an excellent approximation to the energy of the oscillator potential with basic frequency  $a/\sqrt{2m}$ . The considered eigenfunctions of the Mathieu equation expressed in terms of  $x$  instead of  $u$  are in the limit  $a \rightarrow \infty$  equal to the eigenfunctions of the oscillator problem in ordinary quantum mechanics (Ref. <sup>14</sup>, Appendix 1, 5 p. 369).

The problem can also be solved by expressing the Hamiltonian  $H = P^2/2m + V$  in terms of  $P$  and  $Q(0)$  for which we have known representations and finding a nonsingular matrix for which  $SHS^{-1}$  is diagonal. One arrives this way at the chain fractions for the eigenvalues of the Mathieu equation.

We have seen that eigenfunctions and eigenvalues of the present oscillator for a suitably chosen potential approximate those of the oscillator prob-

lem in ordinary quantum mechanics. This is not just a coincidence. Consider two Sturm-Liouville problems of this kind, both of the general form:

$$y'' - (q(x) + \lambda)y = 0 \quad (3)$$

which differ in the boundary conditions:

$$y(-\pi) = y(\pi) \quad (3a)$$

for the first where the interval is  $(-\pi, \pi)$  and

$$y(-\infty) = y(\infty) = 0 \quad (3b)$$

for the second.

The extremal principle of the Sturm-Liouville problem<sup>15</sup> states that the  $n$ -th eigenvalue of the problem is equal to the minimum of:

$$E_n = \min \int (y_n'^2 + q(x)y_n^2) dx \quad (4)$$

if the subsidiary conditions:

$$\int y_n y_m dx = \delta_{mn} \quad (4a)$$

are fulfilled for  $n \geq m$  where  $y_m$  is the  $m$ -th eigenfunction of the problem and the prescribed boundary conditions are also fulfilled. Eigenfunctions  $y_n(x)$  and eigenvalues  $E_n$  can be obtained with the help of this variational problem.

In both oscillator problems the functions  $q(x)$  are such that  $q(0) = 0$ , the growth of  $q(x)$  with  $|x|$  in the neighbourhood of  $x=0$  is monotonous and such that  $q(1)$  is very much greater than one. (We choose of course also for the ordinary oscillator the same units as for the presently treated.) Furthermore there exist values  $x_0: |x_0| < 1$  so that the absolute value of the difference of the functions  $q(x)$  of both problems is everywhere smaller than one for  $|x| < |x_0|$  and tends to zero in the limit  $x \rightarrow 0$ .

The above minimum condition for the eigenvalue problem Eq. (4, 4a) implies then that in both problems the eigenfunctions will "crowd" in the same way into as narrow an interval around the origin as possible to avoid larger values of  $q$ . Thus eigenfunctions and eigenvalues should be comparable because the  $q(x)$  are comparable as long as they are small. This relation continues till such values of  $n$  where the orthogonality conditions "force" the eigenfunctions out of the interval where the two  $q$ -functions are nearly equal.

#### IV. The Uncertainty Relation in the discrete Case

The finiteness of the spaces considered in the previous sections poses an upper limit to the un-

certainty of the position. The uncertainty of momentum can become zero when this upper limit is reached. The momentum is then sharp and the product  $\Delta P \Delta Q = 0$ . How has the uncertainty relation to be modified in the case of discrete momentum spectrum in order to consider this fact and yet to remain in agreement with ordinary quantum mechanics in the local limit? To analyze this question we ask for the minimum of the product  $\Delta P \Delta Q$  attainable if  $\Delta Q$  assumes a fixed value:  $\Delta Q = A$ . We have then the conditions:

$$\int \psi^* \psi dx = 1 \quad (\text{normalization}), \quad (1a)$$

$$\int \psi^* \psi X dx = \tilde{x} \quad (\text{mean value of position}), \quad (1b)$$

$$\int \psi^* \psi X^2 dx = A^2, \quad (1c)$$

$$1/i \int \psi^* \psi' dx = \tilde{p} \quad (\text{mean value of momentum}). \quad (1d)$$

The integration extends in case of Euclidian space and the boundary conditions:  $\psi(-\infty) = \psi(\infty) = 0$  from  $-\infty$  to  $\infty$ . The position operator  $X$  is to be replaced by  $x$  in this case. The integration over the closed space extends from  $-\pi$  to  $\pi$  and the boundary conditions  $\psi(-\pi) = \psi(\pi)$ . The position operator  $X$  is replaced here by  $\sin x$ . One may assume without restriction of generality that the mean value of position is situated at the origin of our coordinate system:  $\tilde{x} = 0$ . We therefore do not consider anymore condition (1b); its consequences can be eliminated by a trivial transformation.

We want the function for which:

$$\Delta P^2 = - \int \psi^* \psi'' dx - p^2$$

is minimum under the rest of the above subsidiary conditions. We use multipliers  $\lambda, \mu, \nu, \varrho$  and impose:

$$\delta \left[ \int \{ -\psi^* \psi'' - i \lambda \psi^* \psi' + (\mu X^2 + \nu) \psi^* \psi \} dx - \tilde{p}^2 - \lambda \tilde{p} - \mu A^2 - \nu + \varrho (\text{bound. cond}) \right] = 0. \quad (2)$$

Variation w.r.t.  $\tilde{p}$  yields:  $2\tilde{p} + \lambda = 0$  and variation w.r.t.  $\psi^*$  yields the Euler equation:

$$\psi'' - 2i\tilde{p}\psi' - (\mu X^2 + \nu)\psi = 0. \quad (2a)$$

Substitution for the position operator  $X$  is performed as indicated before. Setting:

$$\psi = y e^{i\tilde{p}x} \quad (2b)$$

one obtains the equation for  $y$

$$y'' - (\mu X^2 + \nu + \tilde{p}^2)y = 0. \quad (2b')$$

One recognizes that for  $X = x$  this is the equation of the harmonic oscillator and for  $X = \sin x$  it is the Mathieu equation.

The extremum principle of the Sturm-Liouville problem (III.4) shows that for fixed  $\Delta X = A$  the minimum of  $\Delta P$  is obtained for the lowest eigenvalue. The corresponding eigenfunction is in the Euclidian case the lowest harmonic oscillator function and in the other case the lowest Mathieu function. The latter according to section III is approximated in the limit  $A \rightarrow 0$  by the former. The lowest eigenfunction of the harmonic oscillator is

however the normalized Gaussian which is well known to be the function  $\psi(x)$  of minimum uncertainty. The uncertainty relation of the quantum mechanics in the closed space is therefore in the limit of vanishing  $A$  the same as in Euclidian space. The other extreme  $A \approx \pi$  allows to let  $\mu$  go to zero so that the Mathieu equation has trigonometric functions as solutions and we know that then  $\Delta P = 0$ .

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